HOW TO RELEASE FREGE'S SYSTEM FROM RUSSELL'S ANTINOMY

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Abstract. The conditions for proper definitions in mathematics are given, in terms of the theory of definition, on the basis of the criterions of eliminability and non-creativity. As a definition, Russell's antinomy is a violation of the criterion of eliminability (Behmann, 1931, [1]; Bochvar, 1943, [2]). Following the path of the criterion of non-creativity, this paper develops a new analysis of Comprehension schema and, as a consequence, proof that Russell's antinomy argumentation, despite the words of Frege himself, does not hold in Grundgesetze der Arithmetik. According to Basic Law (III), the class of classes not belonging to themselves is a class defined by a function which can not take as argument its own course of value,

$$\top\left(\forall = \grave{\varepsilon}\left(\top\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \rightarrow \mathfrak{g}(\varepsilon)\big)\right)\right),$$

in other words, the class of classes not belonging to themselves is a class whose classes are not identical to the class itself [6].

Introduction

Recently, a considerable attention has been devoted to the discovery of new ways for repairing Frege's system from the damage produced by Russell's antinomy. All these attempts share a common aptitude in finding new restrictive forms or applications of the comprehension principle [4, 16, 20, 22, 25], either offer a revised interpretation of Basic Law (V) [3].

The present note proposes a completely new way to restore Frege's system, neither restrictive nor revising, rising from the logical inquiry in the theory of definition and the author's investigations in self-reference procedures [7, 9]. Partially in agreement with their constructive attitude, the result we are concerned with draws its first inspiration from Jules-Henri Poincaré's writings about the significance and riskiness of impredicative definitions in mathematics.

> *The fact is that such a procedure is not applicable. Why? Because their definitions are not predicative and contain within such a vicious circle I already mentioned above; not predicative definitions can not be* substituted to defined terms. *In this condition, logistics is no longer sterile: it generates contradictions.* (1902, [21], 211, our translation)

With these words Poincaré was directly referring to Cantor's famous diagonalization argument for reaching more-than-denumerable sets, criticizing the underlying belief of the existence of the actual infinite. Russell's antinomy and Cantor's diagonalization are both self-referring procedures, and by admission of Russell himself his discovery is due to his reflections about Cantor's argument.

> *I was led to the contradiction in the following way. As you, of course know, Cantor proved that there is no greatest number. [...] Now there are concepts whose extension comprises everything; these should therefore have the greatest number. I tried to set up a one-one relation between all objects and all classes; when I applied Cantor's proof with my special relation, I found that the class* $Cls \cap x \ni (x \sim \epsilon x)$ *was left over, even though all classes had already been enumerated. I have already been thinking about this contradiction for a year; I believe the only solution is that function and argument must be able to vary independently.* ([14], 215, XXXV/3 Russell to Frege 24.6.1902)

Cantor's diagonalization and Russell's antinomy are logically connected, and for the historical perspective the latter rises from the former. We shall expose how the logical inquiry regarding the mathematical definitions and the rules for applying substitutions, as intuited by Poincaré, leads to the restore of Frege's system. Nowadays Russell's antinomy is widespread in many several forms throughout logic and mathematics, we shall apply our argument to three different theories. First, in section 1, to Zermelo-Fraenkel set theory (*ZF*) [10], and to a first order set theory close to Frege's system ([15] 76-90), and finally in section 2, directly to Frege's Grundgesetze der Arithmetik [12]. For this last section the redear is required to be familiar with the original notation of Grundgesetze. This paper is the revised and extended version of [8].

1. ZERMELO-FRAENKEL SET THEORY

In Cantor's *naive* set theory a set was thought to be an aggregate, i.e a "collection of elements into a whole of our intuition". This conception appears accordingly symbolized by the principle of comprehension, which can be regarded as a schema originally conceived for defining sets. For this special function, we are legitimate in considering the comprehension principle under the point of view of the theory of definition[1]. On that basis the rules for definitions, established by the criterions of eliminability and non-creativity, state the conditions for proper equivalence, giving some basic restrictions to prevent superimpositions and circularity ([24], 151-173). The rule for defining a *new operation symbol* (or a new individual constant, i.e. an operation symbol of rank zero) is as follows:

[1]We are referring here to the field of logical researches of ancient tradition, which in the last century has seen noteworthy contributions from A. Padoa, J.C.C. Mckinsey, A. Tarsky, E. W. Beth, A. Robinson, W. Craig. It is also worth mentioning how significative the theory of definition was throughout Frege's whole foundation of mathematics [13, 11, 12], even if it was unfortunately neglected, as we will see, in the famous rushed appendix ([12], II 253-265). The principle of Comprehension results to be formalized for the first time in *Grundgesetze*,

$$\vdash f(a) = a \frown \grave{\varepsilon} f(\varepsilon),$$

but its role in defining *new objects* seems not to be sufficiently focussed by Frege ([12], I 73, 75, 117). All that appears to be comprehensible for that time, as the modern theory of definition was at its beginning exactly with Frege and Peano. Remained in the twilight for years, the first notable result investigating the meaning of the introduction of *new symbols* for the interplay between mathematical definition and logical deduction, could be viewed as an outcome of the questions raised in Frege–Peano correspondence [19].

r_1 An equivalence E introducing a new n-place operation symbol O is a proper definition in a theory if, and only if, E is of the form

$$O(x_1 \ldots x_n) = y \leftrightarrow \psi,$$

and the following restrictions are satisfied: *(i)* $x_1 \ldots x_n$, y are distinct variables, *(ii)* ψ has no free variables other than $x_1 \ldots x_n, y$, *(iii)* ψ is a formula in which the only non-logical constants are primitive symbols and previously defined symbols of the theory, and *(iv)* the formula $\exists! y\psi$ is derivable from the axioms and preceding definitions of the theory.

With condition *(iv)* we are required to have a preceding theorem which guarantees that the operation is uniquely defined. If the restriction on the uniqueness is dropped then a contradiction can be derived ([24], 159). The self-referring characteristic of Russell's antinomy will turn out to be a procedure which forces a set to be twice defined, we are dealing with a set whose elements belong and do not belong to the set itself, dropping thus the above uniqueness condition.

As it is well-known, in *ZF* when defining a set, the uniqueness for the comprehension axiom schema is ensured by the axiom of extensionality. Let us introduce to them both,

$$(C) \qquad \textit{Comprehension} \;\; \forall z_1 ... \forall z_n \exists y \forall x (x \in y \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ is any formula in *ZF*, $z_1, ..., z_n$ are the free variables of $\varphi(x)$ other than x, and y is not a free variable of $\varphi(x)$,

$$(E_1) \qquad \textit{Extensionality} \;\; \forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \leftrightarrow x = y].$$

Extensionality, without any additional axioms, implies that for every condition $\varphi(x)$ on x (in *Comprehension*) there exists "at most" one set y which contains *exactly* those elements x which fulfil the condition $\varphi(x)$. In other words, if there is a set y such that $\forall x(x \in y \leftrightarrow \varphi(x))$, y is unique. It can be shown as follows. If y' is also such, i.e. $\forall x(x \in y' \leftrightarrow \varphi(x))$, then we have, obviously $\forall x(x \in y' \leftrightarrow x \in y)$, and then by *Extensionality*, $y' = y$. ([10] 31, [18] 7, [5] 67).

Usually Russell's antinomy argumentation is presented as follows.

RA *There exists no set which contains exactly those elements which do not contain themselves, in symbols* $\neg \exists y \forall x (x \in y \leftrightarrow x \notin x)$.
Proof. By contradiction. Assume that y is a set such that for every element x, $x \in y$ if and only if $x \notin x$. For $x = y$, we have $y \in y$ if and only if $y \notin y$. Since, obviously, $y \in y$ or $y \notin y$, and as we saw, each of $y \in y$ and $y \notin y$ implies the other statement, we have both $y \in y$ and $y \notin y$, which is a contradiction ([10], 31).

This proof holds for sure as a first order theorem, as indeed in first order logic

$$\vdash \neg \exists y \forall x (x \in y \leftrightarrow x \notin x),$$

and it can affect comprehension principle when it is regarded individually or in itself

$$(C) \rightarrow \exists y \forall x (x \in y \leftrightarrow x \notin x) \vdash,$$

but it does not hold in any system which applies both *Comprehension* and *Extensionality*

$$(C) \cup (E_1) \vdash \neg \forall x (x \in y \leftrightarrow x \notin x).$$

Indeed we can show that Russell's antinomy does not affect a first order set theory, since by *Extensionality* the above proof RA can not be accomplished. If as an example of *Comprehension* we define

$$x \in y \text{ if and only if } x \notin x$$

then by *Extensionality* we obtain always (detailed proof in the following of this section and in section 2)

$$(*) \qquad (x \in y \leftrightarrow x \notin x) \to \neg(x = y),$$

so it is never the case that $x = y$. This is an applicative case of the uniqueness condition. Indeed a special case because it involves negation and therefore complementation. For it, the above inference RA can not be accomplished since "$x = y$" can not be assumed and "$y \in y$ if and only if $y \notin y$" is not derived. In other terms, for a simple first order rule from

$$\forall x(x \in a \leftrightarrow x \notin x)$$

we can yield

$$(a \in a \leftrightarrow a \notin a),$$

but this rule is not applicable during a reasoning of set theory where, by *Extensionality*, the formula

$$(**) \qquad (x \in a \leftrightarrow x \notin x) \to \neg(x = a)$$

can always be derived.

Two sets are equal when and only when they have the same elements, and the equality between sets must fulfill all the requirements of the relation of identity, namely, reflexivity, every set has the same elements as itself, and substitutivity of identity, $x = y \to (P(x) \leftrightarrow P(y))$ (for x and y any set and P any property). Actually the substitutivity of identity is a version of the Principle of the Indiscernibility of Identicals, one of the consequences of Frege's Basic Law (III) (see section 2). In set theory it doesn't follow directly from the relation of coexstensiveness, and, whether axiom or hypothesis, a principle must be stated that two coextensive sets share all their properties and are therefore equal. The principle of extensionality says in effect that two sets are identical if and only if they have the same elements, in symbols (E_1). Let us recall also a further first order schema as follows,

$$(E_2) \qquad \textit{Extensionality} \;\; \forall x \forall y[x = y \to (\phi(x, x) \leftrightarrow \phi(x, y))],$$

where $\phi(x, y)$ is obtained from $\phi(x, x)$ by replacing y for zero, one or more occurrences of x in the wff $\phi(x, x)$, and y is free for x in all occurrences of x which it replaces ([15] 79). Under the logical point of view (E_2) is, as respect to (E_1), the integral schema, quite close to Frege's Basic Law (III). (E_2) yields

$$\forall x \forall y[x = y \to \forall z(z \in x \leftrightarrow z \in y)]$$

([15] 139), which holds usually in ZF together with the further statement

$$\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \to x = y],$$

either as axiom ([10] 27), or theorem ([15] 141). Any derivation of $(*)$ in ZF requires necessarily both statements, so that we adopted directly (E_1) ([17] 109). As in Frege's Grundgesetze, see section (2), $(*)$ follows easily from (E_2), whereas the derivation of $(*)$ from (E_1) is bound to the methodological concerns of ZF. We shall expose both the cases.

Let us begin with the first,

$$(I) \qquad x = y \to (x \in x \leftrightarrow x \in y)$$

is an instance of (E_2) where $\phi(x, x)$ is $x \in x$ and $\phi(x, y)$ is $x \in y$, so that $x \in y$ is obtained by substituting x with y in $x \in x$. We notice that the restriction on y in (E_2) is fulfilled so that (I) follows logically from (E_2). (I) yields by $(p \to q) \leftrightarrow (\neg q \to \neg p)$and $(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$

$$(II) \qquad \neg(x \in y \leftrightarrow x \in x) \to \neg(x = y)$$

and then,$\neg(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow \neg q)$,

$$(III) \qquad (x \in y \leftrightarrow x \notin x) \to \neg(x = y),$$

i.e. $(*)$ QED.

In *ZF* $(*)$ can be derived easily from (E_1) too, with the exception of the methodological limitations about illegitimate self-referring.

> *In most cases when we use variables as metamathematical variables for variables we shall assume, tacitly, that different variables stand for different variables. E.g. the set of all statements* $\forall z(z \in x \leftrightarrow z \in y)$ *is also assumed to contain the statement* $\forall u(u \in w \leftrightarrow u \in v)$, *but not to contain the statement* $\forall x(x \in x \leftrightarrow x \in y)$. *In some other cases we do not insist that different variables stand for different variables.* ([10] 21)

As well-known set theorists use methodically such tacit assumptions in *ZF* with the purpose of preventing self-contradictory procedures from arising. But our aim is actually just to show that Russell's antinomy does not follow when *Extensionality* is taken suitably into account. We shall place therefore the above tacit assumptions in stand by, and we shall keep in enquiring instead into the restrictions established by the theory of definition.

Furthermore, let us add, that if we allow expressions of totalities and self-reference in *Comprehension* schema there is no reason to prevent the same expressions within *Extensionality* schema.

Accordingly, in *ZF*

$$(x \in y \leftrightarrow x \in x) \leftrightarrow x = y,$$

is an instance of (E_1), hence

$$\neg(x \in y \leftrightarrow x \in x) \to \neg(x = y),$$

and finally

$$(x \in y \leftrightarrow x \notin x) \to \neg(x = y),$$

i.e. $(*)$, QED. Whenever (C) yields $\forall x(x \in a \leftrightarrow x \notin x)$ by $(*)$ we have $(**)$ too.

We can now state that Russell's antinomy argumentation does not hold in a first order system based on *Comprehension* and *Extensionality*, because it violates the restriction on the uniqueness established by extensionality. In defining a set containing exactly those elements which do not contain themselves we define a set which members can not be considered to be identical to the set itself, either we should have a set defined twice, as a set which contains and does not contain itself. This would be to define a set and its complement as being exactly the same identical set, and in detail to violate the criterion of non-creativity and its consequent criterion of relative consistency ([24], 155). From the logical point of view Russell's antinomy turns out to be just a partial argument. Although set theory disposes of

the symbols to mention, or to express, the existence of sets containing exactly those elements which do not contain themselves, yet, when the argument reaches its own whole representation, no contradiction can be derived. All that turns out to be amazing especially in Frege's Grundgesetze, since the subcomponent "$\top(\forall = \forall)$", which prevent the contradiction in Frege's *way out* ([12], II 262-263), can be derived simply by Basic Law (III), namely without Frege's restrictions. The so called "Frege's *way out*" is in the relative literature the shortening for the original Frege's attempt to repair his system, by means of (V'b) and (V'c). As well-known the amended axiom blocks the derivation of Russell's antinomy, but it is not formally satisfactory ([23] 165-170). On the contrary the solution we propose here seems to be formally faultless.

2. FREGE SYSTEM

We are now ready to reconsider the appendix of Grundgesetze der Arithmetik ([12], II 253-265) in accordance with what was previously stated. We shall prove that despite the words of Frege himself Russell's antinomy can not be derived.

In the following, Frege's symbolism is fully adopted, except for the modern symbol of implication. First let us look at the deduction involved with function $\zeta \frown \xi$.

> *... Now let us see how the matter turns out if we make use of our sign "$\frown$". Here "$\grave{\varepsilon}(\top \varepsilon \frown \varepsilon)$" will occur in place of "$\forall$". If in our proposition*

$$F(a \frown \grave{\varepsilon} f(\varepsilon)) \;\to\; F(f(a)) \tag{82}$$

> *we take*

i) "$\top \xi \frown \xi$" *for* "$f(\xi)$",

ii) "$-\!\!\!-\xi$" *for* "$F(\xi)$",

iii) and "$\grave{\varepsilon}(\top \varepsilon \frown \varepsilon)$" *for* "$a$",

> *then we obtain*

$$\grave{\varepsilon}(\top \varepsilon \frown \varepsilon) \frown \grave{\varepsilon}(\top \varepsilon \frown \varepsilon) \to \top \grave{\varepsilon}(\top \varepsilon \frown \varepsilon) \frown \grave{\varepsilon}(\top \varepsilon \frown \varepsilon), \tag{θ}$$

> *from which by (Ig) there follows*

$$\top \grave{\varepsilon}(\top \varepsilon \frown \varepsilon) \frown \grave{\varepsilon}(\top \varepsilon \frown \varepsilon). \tag{ι}$$

> *Making the same substitutions in proposition*

$$F(f(a)) \;\to\; F(a \frown \grave{\varepsilon} f(\varepsilon)) \tag{77}$$

> *we obtain*

$$\top \grave{\varepsilon}(\top \varepsilon \frown \varepsilon) \frown \grave{\varepsilon}(\top \varepsilon \frown \varepsilon) \to \grave{\varepsilon}(\top \varepsilon \frown \varepsilon) \frown \grave{\varepsilon}(\top \varepsilon \frown \varepsilon), \tag{κ}$$

> *from which together with (ι) there follows*

$$-\!\!\!-\grave{\varepsilon}(\top \varepsilon \frown \varepsilon) \frown \grave{\varepsilon}(\top \varepsilon \frown \varepsilon), \tag{λ}$$

> *which contradicts (ι). Therefore at least one of the two (77) and (82) must be false, and therefore proposition*

$$\vdash f(a) = a \frown \grave{\varepsilon} f(\varepsilon), \tag{1}$$

> *also, from which they both follow. A look at the derivation of (1) in §55 of our first volume shows that there too use is made of (Vb). Thus suspicion is directed at this proposition here as well.* ([12], II 257)

Within Frege's system the proposition

$$\vdash a = b \;\rightarrow\; (g(a) = g(b))$$

(IIIh, [12], I 66-67) can be achieved by basic law

(III) $$\vdash f(a = b) \rightarrow f(-\!\!\overset{\mathfrak{g}}{\smile}\!\!-(\mathfrak{g}(b) \rightarrow \mathfrak{g}(a))),$$

as well as the proposition $(\neg\, g(a) = g(b)) \;\rightarrow \neg\,(a = b)$.

In detail from

$$\neg\, g(a) \;\rightarrow\; (g(b) \;\rightarrow \neg\,(a = b))$$

(IIIb, ibid. 66-67), and $g(\xi) = (\neg\, g(a) = g(\xi))$ we obtain

$$\neg\neg\, g(a) = g(a) \;\rightarrow\; ((\neg\, g(a) = g(b)) \;\rightarrow \neg\,(a = b)),$$

and

$$g(a) = g(a) \;\rightarrow\; ((\neg\, g(a) = g(b)) \;\rightarrow \neg\,(a = b)),$$

so that, by $\vdash a = a$ (IIIe, ibid. 66-67),

(2) $$(\neg\, g(a) = g(b)) \;\rightarrow \neg\,(a = b).$$

Let us now reconsider in accordance the above Frege's substitutions in (82) and (77). Firstly, by the substitution *i*) in law (1) we obtain

(3) $$\neg\, a \frown a \;=\; a \frown \grave{\varepsilon}(\neg\, \varepsilon \frown \varepsilon).$$

Then replacing

iv) "$a \frown \xi$" for "$g(\xi)$",

v) and "$\grave{\varepsilon}(\neg\, \varepsilon \frown \varepsilon)$" for "$b$",

in (2) we obtain

(4) $$(\neg\, a \frown a \;= a \frown \grave{\varepsilon}(\neg\, \varepsilon \frown \varepsilon)) \;\rightarrow \neg\,(a = \grave{\varepsilon}(\neg\, \varepsilon \frown \varepsilon)).$$

Therefore by (3) and (4)

(5) $$\neg\,(a = \grave{\varepsilon}(\neg\, \varepsilon \frown \varepsilon)).$$

Now the replacement of "$\grave{\varepsilon}(\neg\, \varepsilon \frown \varepsilon)$" for "$a$" in (3) turns out to be invalid, and consequently Frege's substitution *iii*) in (82) and (77) can not be accomplished. Therefore (θ), (ι), (κ), (λ) and the contradiction can no longer be derived[2], QED.

Our proof shows that Frege's *way out* of negating for any function the possibility to take as argument its own course of value ([12], II 262), as in

(V'b) $$\vdash (\grave{\varepsilon} f(\varepsilon) = \grave{\alpha} g(\alpha)) \rightarrow (\neg\,(a = \grave{\varepsilon} f(\varepsilon)) \rightarrow f(a) = g(a))$$

and

(V'c) $$\vdash (\grave{\varepsilon} f(\varepsilon) = \grave{\alpha} g(\alpha)) \rightarrow (\neg\,(a = \grave{\alpha} g(\alpha)) \rightarrow f(a) = g(a)),$$

turns out to be useless and too restrictive. Frege's Basic Law (III) already rules out the cases of those functions which would lead to contradiction taking as argument its own course of value, without need to exclude all the other cases of self-referring.

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[2] Compare with Frege's amendment, ($1'$ ([12] 264-265)

Indeed there can be cases of functions which argument can be its own course of value without yielding any contradiction, so that there is no reason for excluding them a priori. To clarify all that, let us recall Frege's passage after the introduction of (V'b) and (V'c).

> *Let us now convince ourselves that the contradiction that arose earlier between the proposition (β) and (ϵ) is now avoided. We proceed as we did in the derivation of (β), using (V'c) instead of (Vb). As before, let*
>
> $*$) "$\forall$" *abbreviate* "$\grave{\varepsilon}\Big(\neg\overset{\mathfrak{g}}{\smile}\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big)$".
>
> *By (V'c) we have*

$$\vdash \grave{\varepsilon}(-\!\!-f(\varepsilon)) = \grave{\varepsilon}\Big(\neg\overset{\mathfrak{g}}{\smile}\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big) \to$$
$$\Bigg(\neg\Big(\forall = \grave{\varepsilon}\Big(\neg\overset{\mathfrak{g}}{\smile}\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big)\Big) \to$$
$$(-\!\!-f(\forall)) = \neg\overset{\mathfrak{g}}{\smile}\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall)\big)\Bigg).$$

> *Using our abbreviation, we obtain*

$$\vdash \grave{\varepsilon}(-\!\!-f(\varepsilon)) = \grave{\varepsilon}\Big(\neg\overset{\mathfrak{g}}{\smile}\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big) \to$$
$$\Bigg(\neg(\forall = \forall) \to$$
$$(-\!\!-f(\forall)) = \neg\overset{\mathfrak{g}}{\smile}\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall)\big)\Bigg),$$

> *which is obviously true, because of the subcomponent* "$\neg(\forall = \forall)$" *and on that very account can never lead to a contradiction.* ([12], II 262-263, for (β and (ϵ see ibid. 256)

We show now that by basic laws (V) and (III) we can achieve immediately

$$\neg(\forall = \forall)$$

and

$$\neg\Big(\forall = \grave{\varepsilon}\Big(\neg\overset{\mathfrak{g}}{\smile}\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big)\Big)$$

without applying V'c.

By the definition of the identity sign ([12], I 11), the notation for the course of value ([12], I 15) and $*$), the symbols "$\forall$" and "$\grave{\varepsilon}\Big(\neg\overset{\mathfrak{g}}{\smile}\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big)$" denote the same object, and, if from basic law

(V) $$\vdash (\grave{\varepsilon}g(\varepsilon) = \grave{\alpha}f(\alpha)) = (\overset{\mathfrak{a}}{\smile}\, g(\mathfrak{a}) = f(\mathfrak{a}))$$

we draw

$$\left(\grave{\varepsilon}\Big(\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big) = \grave{\varepsilon}(-\!\!-f(\varepsilon))\right) = \left(\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall)\big) = (-\!\!-f(\forall))\right), \tag{6}$$

which means $\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall)\big) = (-\!\!-f(\forall))$ is true if and only if "$\grave{\varepsilon}\Big(\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big)$" and "$\grave{\varepsilon}(-\!\!-f(\varepsilon))$" refer to the same object, then consequently

$$\left(\grave{\varepsilon}\Big(\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big) = \forall\right) = \left(\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall)\big) = (-\!\!-f(\forall))\right). \tag{7}$$

Now substituting

vi) "$-\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \xi \to \mathfrak{g}(\xi)\big)$" for "$f(\xi)$"

in (7) we obtain

$$\left(\grave{\varepsilon}\Big(\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\big)\Big) = \forall\right) = \left(\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall) = -\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall)\big)\right). \tag{8}$$

We replace then

vii) "$-\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \xi \to \mathfrak{g}(\xi)\big)$" for "$g(\xi)$",

viii) "$\forall$" for "a",

ix) "$\forall$" for "b"

in (2) yielding

$$\left(\top\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall)\big) = -\!\overset{\mathfrak{g}}{\smile}\!-\big(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \forall \to \mathfrak{g}(\forall)\big)\right) \to \top(\forall = \forall). \tag{9}$$

Hence for $*$), (8) and (9)

$$\top(\forall = \forall). \tag{10}$$

Let us regard everything by means of a different substitution. By (V) we can yield

$$(11)\quad \left(\grave{\varepsilon}\left(\top\!\overset{\mathfrak{g}}{\smile}\!\left(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\right)\right) = \forall\right) =$$

$$\left(\top\!\overset{\mathfrak{g}}{\smile}\!\left(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \grave{\varepsilon}\left(\top\!\overset{\mathfrak{g}}{\smile}\!\left(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\right)\right) \to \right.\right.$$

$$\left.\left.\mathfrak{g}\left(\grave{\varepsilon}\left(\top\!\overset{\mathfrak{g}}{\smile}\!\left(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\right)\right)\right)\right) = (-\!\!-f(\forall))\right).$$

Then for *vi*) in (11), and *vii*), *ix*) and

$$x)\qquad \text{“}\grave{\varepsilon}\left(\top\!\overset{\mathfrak{g}}{\smile}\!\left(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\right)\right)\text{” for “}a\text{”},$$

in (2) we achieve

$$(12)\qquad \top\!\left(\forall = \grave{\varepsilon}\left(\top\!\overset{\mathfrak{g}}{\smile}\!\left(\grave{\varepsilon}(-\!\!-\mathfrak{g}(\varepsilon)) = \varepsilon \to \mathfrak{g}(\varepsilon)\right)\right)\right).$$

Accordingly, the above mentioned (β) and (ϵ) can not be drawn, without applying (V'c) or (V'b), and no contradiction can be derived. We have thus proved that that special self-reference procedure which is Russell's antinomy does not damage Frege's system, QED.

Ending Notes. Following Frege's *way out*, all subsequent literature states the necessity to assert the comprehension schema in some restricted form such as

$$a \in \grave{\varepsilon}f(\varepsilon) \leftrightarrow (a \neq \grave{\varepsilon}f(\varepsilon) \wedge f(a)),$$

(see for example [23], 168). On the bases of our enlightenment the uniqueness established by extensionality turns out to be sufficient to prevent the inference of Russell's antinomy, preserving in addition the assumption of sound self-reference procedures. As just shown we attain (10) and (12) as consequences of Basic Law (III), which means that the class of classes not belonging to themselves is defined by a function which can not take as argument its own course of value, i.e. it is a class whose classes are not identical to the class itself.

It would be worthless mentioning all the well-known attempts to amend the foundation of mathematics from the supposed damage to the principle of comprehension, like for example Russell's theory of types, Quine's New Foundation, von Neumann-Bernays system of set theory, and so on. We conclude simply observing that in our perspective the necessity of a restriction like Zermelo's *Aussonderung* ([10], 36) appears to be doubtful. Why define a Axiom for 'separating' or 'selecting' those members which fulfil condition $\varphi(x)$, in *Comprehension*, if by *Extensionality* $\exists! y \forall x (x \in y \leftrightarrow \varphi(x))$?

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